Reduced-Order, Multivariable Models with Pure Time Delays

Marshall's method of model reduction (1966) neglects the modes of the high order system which contribute little to the overall system response. This approach is extended to handle complex and/or repeated eigenvalues and is then modified so that pure time delays can be included in the control variables of the reduced order model. Open- and closed-loop examples show that the inclusion of the time delays tends to compensate for the eliminated modes and leads to better agreement between the high and low order models.

ZENTA IWAI
Department of Mechanical Engineering
Kumamoto University
Kumamoto, 860 Japan

D. G. FISHER
Department of Chemical Engineering
University of Alberta
Edmonton, Alberta, Canada T6G 2G6

and D. E. SEBORG
Department of Chemical and Nuclear Engineering
University of California
Santa Barbara, CA 93106

SCOPE

Most modern control techniques require that a suitable mathematical model of the process be available. Frequently it is possible to derive a linear, state space model of the following form which adequately defines the system performance over the region of interest

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t)$ is a vector of n *state* variables and $u(t)$ is a vector of r *input* variables. However, in many cases, the number of state variables, n, is greater than required, or convenient, for control and simulation purposes. Therefore, over the past few years a number of *model reduction* techniques have been developed, several of which are referenced later in this paper. The primary objective of model reduction, in this paper, is to produce a model with the form of Eq. 1 but of order $n_1 < n$. Secondary objectives are to preserve, or guarantee, certain properties in the low order model, e.g. that the steady-state values of the n_1 state variables in the low order model agree with the corresponding values of the original, high order system.

In 1966 Marshall proposed partitioning the state vector into two parts, such that $x = [x_1,x_2]^T$, and then eliminating x_2 by neglecting the dynamics of the corresponding modal variables. For state space models with distinct eigenvalues, his method led to a reduced order model with the following properties:

1) Order, $n_1 < n$ with the form defined by Eq. 1.

2) The state vector, $x_1(t)$, contains the elements specified by the designer and these elements retain their original physical meaning.

3) The steady state values of $x_1(t)$ agree with those of the original high order model.

4) When the dynamics (modes) associated with $x_2(t)$ are significantly "faster" than those of $x_1(t)$, then the reduced order model is a good approximation.

5) The eigenvalues of the reduced order system are a subset of the eigenvalues of the original high order system.

This paper extends Marshall's method so that it handles systems with complex and/or repeated eigenvalues. It also shows how the agreement between the high and low order models can be improved by introducing pure time delays into the control variables of the reduced order model. Several approaches for determining the numerical value(s) of the time delay(s) are also proposed and are easy to visualize and understand. The application of the proposed design method and its effectiveness are illustrated by numerical examples.

CONCLUSIONS AND SIGNIFICANCE

This paper shows how the modal approach to model reduction, first proposed by Marshall (1966), can be extended to handle high order systems with repeated and/or complex eigenvalues. Such models arise in practice, for example in the modelling of the evaporator at the University of Alberta (Fisher and Seborg, 1976), and hence the extension is of practical value as well as theoretical interest.

This paper also shows that the agreement of the state variables, $x_1(t)$, in the reduced order model with the corresponding states in the high order model, can be improved by including one or more time delays in the control variable(s) of the reduced order model. The improvement obtained depends on both the number and numerical value(s) of the time delays that are introduced. Several methods of determining what time delays should be included are proposed, illustrated, and evaluated. The selection of the time delays to be included is usually based on the properties and/or simulated responses of the high order model. However, the time delay values can also be empirically

"tuned" to provide the best agreement possible with responses of the actual process. Simulated responses from a reduced (third) order model of the double effect evaporator at the University of Alberta were significantly closer to the responses of the corresponding states in the original fifth order model when pure time delays were included.

The inclusion of time delays in the low order model can be interpreted as simply a formal extension to multivariable systems of the common practice of including pure time delays in models of single-input, single-output systems to compensate for higher order dynamics that are neglected during the derivation of the (simplified) model. For example a first order transfer function of the form $Ke^{-\tau s}/(\tau_1 s + 1)$ is often used to approximate the response of a higher order system that may not contain a pure time delay.

In general, the inclusion of pure time delays in a model makes it more difficult to apply control design techniques. However, a reduced order model incorporating time delays was used as the basis for the design of a decoupled, closed-loop system and the results presented in Figure 5 show that significantly better results are obtained when time delays are included in the reduced order model.

In summary, it can be said that the familiar model approach to model reduction has been extended in scope and improved in accuracy.

INTRODUCTION

Marshall's modal approach to model reduction (1966) is just one of several different approaches that have been reported in the literature (cf. review articles by Bosley and Lees (1972) and by Decoster and Von Cauwenberge (1976)). Other approaches are based on expansion of the system transfer function (Shamash, 1975; Chen, 1974; Hsia, 1972; Chen and Shieh, 1968; Parthasarathy et al., 1983); singular perturbation theory (Kokotovic and Sannuti, 1968; Kung, 1978; Moore, 1981); or minimization of a quadratic function of the error between the reduced and high order models (Meier and Luenberger, 1967; Sinha and Bereznai, 1971; Tung and Edgar, 1979; Riggs and Edgar, 1974). The least squares model reduction method presented by Edgar (1975) can be extended to handle time delays. Other modal reduction methods have also been proposed. Chidambara and Davison (1967) suggested a different approach, but Graham (1968) showed that it was equivalent to Marshall's method. Davison (1966) and Nicholson (1964) presented another modal method which can result in better agreement between the dynamics of the high and reduced order models but does not guarantee agreement at the final steady state. Wilson (1974) and Wilson et al. (1972, 1973, 1974) compare several model reduction methods and used the modal approach for the reduction of open loop models as well as for reduction of feedback control laws. More recently the problem of generating reduced-order models of large space structures has motivated the development of a number of new methods, e.g. Moore (1981), Shokoohi et al. (1983), Jonckheere and Silverman (1983). Reduction techniques have also been applied to observers, e.g. Inoue and Iwai (1984).

This paper deals specifically with extensions of Marshall's modal method for model reduction.

MARSHALL'S MODEL REDUCTION METHOD

Consider the following stable linear, time-invariant system,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x = R^n$ and $u = R^r$. Equation (1) is referred to as the *high order model.*

Using the similarity transformation

$$x(t) = Mz(t) \tag{2}$$

where M is the modal matrix of A such that $M^{-1}AM = \Lambda$, Eq. 1 becomes:

$$\dot{z}(t) = \Lambda z(t) + M^{-1}Bu(t) \tag{3}$$

In Marshall's method it is assumed that the eigenvalues, λ, of A are distinct and negative so Λ = diagonal $\{-\lambda_1 \ldots -\lambda_n\}$.

It is now assumed that the state vector, x, can be re-ordered if necessary and partitioned into two parts such that $x = [x_1, x_2]^T$. The vector x_1 of dimension n_1 contains the states that are to be retained in the reduced order model and is normally associated with the dominant dynamics of the high order model. (Iwai and Kubo, 1979, show that for the "best" reduced order system it is necessary to consider the input gain matrix as well as the absolute value of the eigenvalues.) If z is similarly partitioned, then using Eqs. 1-3 the dynamic behavior of x_1 can be described by

$$\dot{x}_1(t) = M_1\Lambda_1 M_1^{-1}x_1(t) + \Gamma_1 z_2(t) + B_1 u(t) \tag{4}$$

where

$$\dot{z}_2(t) = \Lambda_2 z_2(t) + \Gamma_2 u(t) \tag{5}$$

where B_1 is an $n_1 x r$ partition of B; M is partitioned into four submatrices $M_1 \ldots M_4$ which are $(n_1 x n_1)$, $(n_1 x n_2)$, $(n_2 x n_1)$ and $(n_2 x n_2)$ respectively; V_1 is an $n_1 x n_1$ partition of $V = M^{-1}$; Λ_1 is an $n_1 x n_1$ partition of Λ; and

$$\Gamma_1 = M_2\Lambda_2 - M_1\Lambda_1 M_1^{-1}M_2$$

$$\Gamma_2 = V_3 B_1 + V_4 B_2$$

Marshall's method of model reduction assumes that the eigenvalues Λ_2 are nondominant, i.e. that $\dot{z}_2(t) = 0$ for $t > 0^+$, so that Eq. 5 becomes

$$z_2(t) = -\Lambda_2^{-1}\Gamma_2 u(t) \tag{6}$$

Equations 4 and 6 define the *reduced order model* generated by Marshall's method (1966). If the eigenvalues of the high order model are distinct, and if there is good separation between the eigenvalues Λ_1 and Λ_2, then Eqs. 4 and 6 can provide a good approximation to the behavior of x_1 in Eq. 1. Note that a number of variations can be generated by using different approximations for $z_2(t)$ in Eq. 4.

EXTENSIONS TO MARSHALL'S MODEL REDUCTION METHOD

In this section Marshall's method will be extended to handle repeated and/or complex eigenvalues. Time delays will also be introduced into the control variables of the reduced order model to help compensate for the neglected dynamics of $z_2(t)$.

For the general case, it should be noted that Eq. 5 can be written in Jordan canonical form as:

$$\dot{z}_2 = \begin{bmatrix} \dot{z}_{21} \\ \cdots \\ \dot{z}_{2l} \\ \cdots \\ \dot{z}_{2m} \end{bmatrix} = \begin{bmatrix} \Lambda_{21} & 0 & 0 \\ 0 & \Lambda_{2l} & 0 \\ 0 & 0 & \Lambda_{2m} \end{bmatrix} \begin{bmatrix} z_{21} \\ \cdots \\ z_{2l} \\ \cdots \\ z_{2m} \end{bmatrix} + \begin{bmatrix} \Gamma_{21} \\ \cdots \\ \Gamma_{2l} \\ \cdots \\ \Gamma_{2m} \end{bmatrix} u \tag{7}$$

where vector z_{2l}, of dimension n_{2l}, is a subset of z_2 such that $\Sigma_{l=1}^{m} n_{2l} = n_2$. The lth partition of z_2 from Eq. 7 can be written

$$\dot{z}_{2l}(t) = \Lambda_{2l} z_{2l}(t) + \Gamma_{2l} u(t) \tag{8}$$

and applying the Laplace transform, assuming zero initial conditions, yields

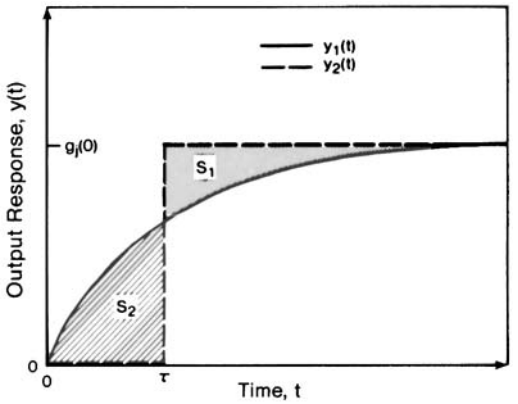

Figure 1. Delayed step response $y_2(t)$ approximates response of high order model $y_1(t)$. (τ is a design parameter.)

$$z_{2l}(s) = (sI - \Lambda_{2l})^{-1}\Gamma_{2l}u(s). \tag{9}$$

There are three different cases corresponding to the three possible structures for Λ_{2l}

Case I: distinct eigenvalues

$$\Lambda_{2l} = \text{diag}\,\{-\lambda_1 \ldots -\lambda_{n2l}\} \tag{10}$$

Case II: repeated eigenvalues

$$\Lambda_{2l} = \begin{bmatrix} -\lambda & 1 & 0 \ldots \\ \ldots & & \\ \ldots & 0 & -\lambda \; 1 \\ \ldots & 0 & 0 \; -\lambda \end{bmatrix} \tag{11}$$

Case III: complex eigenvalues $(-\alpha + \beta i)$

$$\Lambda_{2l} = \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix} \tag{12}$$

Each of these cases will be examined separately with the objective of finding an approximation to the vector z_{2l} in Eq. 7 which is *not* a function of x_2. Substitution of the resulting approximation for z_2 into Eq. 4 would then yield a *reduced order model.* Iwai and Kubo (1979) discuss how to select which eigenvalues to eliminate to insure that the integral of the square of the error (ISE) between the step responses of the original and the reduced order models is minimized.

Case Ia: A Block of Distinct Eigenvalues

Substitution of Eq. 10 into Eq. 9 gives

$$z_{2l}(s) = G(s)\Gamma_{2l}u(s) \tag{13}$$

where $G(s) = \text{diag}\,\{(s + \lambda_j)^{-1}\} j = 1 \ldots n_{2l}$.

The following approximation is now introduced,

$$g_{jj}(s) = (s + \lambda_j)^{-1} \approx g_{jj}(0)e^{-\tau_j s} \tag{14}$$

where $g_{jj}(0)$ is the steady state gain found by applying the final value theorem to $g_{jj}(s)$ and τ_j is a parameter to be chosen by the system designer (see Figure 1). Substitution of Eq. 14 into Eq. 13 yields:

$$z_{2l}(s) \approx \sum_{j=1}^{n2l} g_{jj}(0)\hat{\Gamma}^j_{2l}\{e^{-\tau_j s}u(s)\} \tag{15}$$

or, by substituting for $g_{jj}(0)$ in terms of λ_j and taking the inverse Laplace transform

$$z_{2l}(t) \approx -\Lambda_{2l}^{-1} \sum_{j=1}^{n2l} \hat{\Gamma}^j_{2l}u(t - \tau_j) \tag{16}$$

where in both equations the $n_2 x r$ matrix

$$\hat{\Gamma}^j_{2l} = \begin{bmatrix} 0 \\ \ldots \\ \bar{\gamma}_j \\ 0 \\ \ldots \end{bmatrix} \quad \text{where } \bar{\gamma}_j = j\text{th row of } \Gamma_{2l},\ j = 1 \ldots n_{2l}$$

If all of the eigenvalues associated with z_2 are distinct, then Eq. 16 becomes

$$z_2(t) \approx -\Lambda_2^{-1} \sum_{j=1}^{n2} \hat{\Gamma}^j_2 u(t - \tau_j) \tag{17}$$

and the reduced order model consists of Eqs. 4 plus 17. If the time delays in Eq. 17 are set equal to zero, then Eq. 17 reduces to Eq. 6 and the result for Case I reduces to Marshall's lower order model. Selection of the time delays, τ_j, is discussed later.

Case Ib: Introduction of Multiple Time Delays

The approximation defined by Eq. 14 involves the introduction of one time delay for each modal variable. A more accurate approximation can be obtained by using multiple time delays. Thus

$$g_{jj}(s) = (s + \lambda_j)^{-1} \approx g_{jj}(0) \sum_{k=1}^{p_j} a_{jk}e^{-\tau jk s} \tag{18}$$

Proceeding as in the previous case yields

$$z_2(t) \approx -\Lambda_2^{-1} \sum_{j=1}^{n2} \hat{\Gamma}_2 \sum_{k=1}^{p_j} a_{jk}u(t - \tau_{jk}) \tag{19}$$

which is the same as Eq. 17 if $p_j = 1$ and the weighting factor $a_{jk} = 1$.

Case II: A Block of Repeated Eigenvalues

Substitution of Eq. 11 into Eq. 9 gives

$$z_{2l}(s) = \begin{bmatrix} g_1(s) & g_2(s) \ldots g_{n2l}(s) \\ & \ldots \\ & g_2(s) \\ 0 & g_1(s) \end{bmatrix} \Gamma_{2l}u(s) \tag{20}$$

where

$$g_i(s) = (s + \lambda)^{-i} \qquad i = 1 \ldots n_{2l}$$

The following approximation is now introduced

$$g_i(s) \approx g_i(0)\, e^{-\tau j s} \tag{21}$$

Substitution of Eq. 21 into Eq. 20 yields

$$z_{2l}(s) \approx \sum_{j=1}^{n2l} g_j(0)\hat{\Gamma}^j_{2l}e^{-\tau_j s}u(s) \tag{22}$$

which after taking the inverse Laplace transform results in

$$z_{2l}(t) \approx \sum_{j=1}^{n2l} g_j(0)\hat{\Gamma}^j_{2l}u(t - \tau_j) \tag{23}$$

where

$$\hat{\Gamma}^k_{2l} = \begin{bmatrix} \bar{\gamma}_k \\ \bar{\gamma}_{k+1} \\ \ldots \\ \bar{\gamma}_{n2l} \\ \mathbf{0} \end{bmatrix} \quad \text{and } \bar{\gamma}_k = k\text{th row of } \Gamma_{2l}$$
$$0 = (k-1)\, xr \text{ zero matrix}$$

If all of the eigenvalues associated with z_2 are equal then $z_2 = z_{2l}$ and the reduced order model is defined by Eqs. 4 plus 23.

Case III: A Pair of Complex Conjugate Eigenvalues, $(-\alpha \pm i\beta)$.

Substitution of Eq. 12 into Eq. 9 gives

$$z_{2l}(s) = \begin{bmatrix} g_\alpha(s) & g_\beta(s) \\ -g_\beta(s) & g_\alpha(s) \end{bmatrix} \Gamma_{2l}u(s) \tag{24}$$

where

$$g_\alpha(s) = (s + \alpha)/\{(s + \alpha)^2 + \beta^2\}$$
$$g_\beta(s) = \beta/\{(s + \alpha)^2 + \beta^2\}$$

As in the previous cases an approximation is made such that

$$g_\alpha(s) \approx g_\alpha(0)\, e^{-\tau_\alpha s}$$
$$g_\beta(s) \approx g_\beta(0)\, e^{-\tau_\beta s} \tag{25}$$

Substitution of Eq. 25 into Eq. 24 leads, after rearrangement and inverse Laplace transformation, to:

$$z_{2l}(t) \approx g_\alpha(0)\Gamma_{2l}u(t - \tau_\alpha) + g_\beta(0)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\Gamma_{2l}u(t - \tau_\beta) \tag{26}$$

If the eigenvalues associated with z_2 consist of one complex conjugate pair, then $z_2 = z_{2l}$ and the reduced order model consists of Eqs. 4 plus 26.

If the eigenvalues associated with z_2 are a mixture of distinct, repeated, and/or complex values, then z_2 is given by Eq. 7 using Eqs. 26, 23, 19, and/or 17, as appropriate, to approximate z_{2l}, $l = 1 \ldots m$.

SELECTION OF VALUES FOR THE TIME DELAY ELEMENTS

It remains to specify the values of the time delay parameters in Eqs. 17, 19, 23, and/or 26 such that the response of the state variables, x_1, in the *reduced* order model (Eq. 4) approximate that of the corresponding state variables in the high order model (Eq. 1).

Consider the unit step response of a continuous function defined by $y_1(s) = g(s)$ and the response of the *corresponding* function defined by $y_2(s) = g(0)\, e^{-\tau s}$. Assume that the step responses have the form shown in Figure 1 and the objective is to pick the time delay, τ, such that $y_2(t)$ is a good approximation to $y_1(t)$. There are obviously several different criteria that could be used to select the "best" approximation. For example, τ could be selected such that the two areas S_1 and S_2 in Figure 1 were equal, or the combined area $S_1 + S_2$ could be minimized. The former could be achieved by setting the integral of the error $y_1(t) - y_2(t)$ equal to zero, and the latter by minimizing the integral of the absolute error (IAE). This approach to selecting values for the time delays will now be applied to each of the reduced order models developed in the previous section.

Case Ia (z_2 defined by Eq. 17)

The approximation defined by Eq. 14 can be interpreted as shown in Figure 1. The integral-error (IE) approach and the IAE approach give the following values:

$$\text{IE (area } S_1 = S_2 \text{ in Figure 1) } \tau_i = 1/\lambda_i \tag{27}$$
$$\text{IAE (area } S_1 + S_2 \rightarrow \text{minimum) } \tau_j = \ln 2/\lambda_j \tag{28}$$

Thus the time delay given by Eq. 27 is equal to the corresponding time constant.

If all of the eigenvalues associated with z_2 are distinct then there will be n_2 different time delays $\{\tau_j, j = 1 \ldots n_2\}$ associated with each control variable. For applications where a single time delay would be more appropriate, Eq. 17 can be approximated by

$$z_2(t) = -\Lambda_2^{-1}\Gamma_2 u(t - \tau_D) \tag{29}$$

where τ_D is to be specified by the designer.

One obvious approach to specifying τ_D in Eq. 29 is to use the average of the n_2 values required by Eq. 17:

$$\tau_D = \frac{1}{n_2}\sum_{j=1}^{n_2}\tau_j \tag{30}$$

The time delay, τ_D, can also be chosen by considering the input sensitivity of the system as follows. Let the input to the system be $\Delta u = (\Delta u_1, \Delta u_2 \ldots \Delta u_r)$ where $\Delta u_i = \delta$ = an arbitrary constant. The response of the ith element z_{2i} of z_2 is then

$$\frac{\Delta z_{2i}(t_1)}{\delta} = \frac{1}{\lambda_{2i}}\sum_{j=1}^{r}\gamma_{ij}[1 - e^{-\lambda_{2i}t_1}] \tag{31}$$

where t_1 is a design parameter, $i = 1 \ldots n_2$ and γ_{ij} is an element of Γ_2. Equation 31 provides a measure of the sensitivity of z_{2i} to a unit step input and leads to specifications such as:

$$\tau_D = \tau_L \tag{32}$$

where L is equal to the index, i, corresponding to

$$\text{Maximum}_{i=1 \ldots n_2}\left|\frac{\Delta z_{2i}}{\delta}\right|$$

and τ is from Eqs. 27 and 28.

Equation 32 implies that τ_D is equal to the delay associated with the system mode that is most sensitive to a step input. Note that t_1 in Eq. 31 determines the time at which the relative sensitivities are calculated. If $t_1 \rightarrow \infty$ this would imply that the final steady state was particularly important.

Case Ib (z_2 defined by Eq. 19)

The reduced order model which allows for multiple time delays in each control variable can be interpreted by referring to Figure 2. The unit step response of the continuous function defined by $(s + \lambda_j)^{-1}$ is shown as $y_1(t)$. The "staircase" approximation defined by Eq. 18 is shown as $y_2(t)$. The objective is to pick the time delays $\{\tau_{jk}, k = 1 \ldots p_j\}$ such that $y_2(t)$ is the "best" approximation to $y_1(t)$.

In order to pick τ_{jk} and a_{jk} in Eq. 18 we utilize the following IAE criterion where the subscript, j, is dropped for simplicity and the error $y_1(t) - y_2(t) = \Delta y$

$$J = \int_0^{\tau_1}|\Delta y|dt + \int_{\tau_1}^{t_1}|\Delta y|dt + \ldots + \int_{\tau_p}^{\infty}|\Delta y|dt \tag{33}$$

From the conditions:

$$\frac{\partial J}{\partial \tau_k} = 0 \qquad k = 1 \ldots p$$
$$\text{and } \frac{\partial J}{\partial t_k} = 0 \qquad k = 1 \ldots p - 1 \tag{34}$$

we obtain the following relationships

$$2e^{-\lambda\tau k} = e^{-\lambda t_{k-1}} + e^{-\lambda t_k}$$
$$2t_k = \tau_k + \tau_{k+1} \tag{35}$$

Then by setting $e^{-\lambda t}k = \chi_k$ we obtain

$$3\chi_k^2 = \chi_{k-1}\chi_k + \chi_k\chi_{k+1} + \chi_{k+1}\chi_{k-1}, \quad k = 1 \ldots p - 1 \tag{36}$$

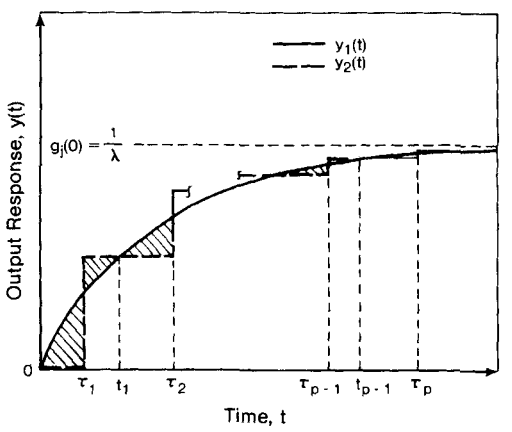

Figure 2. Time delay parameters $\tau_1 \ldots \tau_p$ selected so that the "staircase" function $y_2(t)$ approximates $y_1(t)$.

TABLE 1. TIME DELAY VALUES FOR EQ 21

j	$\bar{\alpha}$	
1	0.693	
2	1.678	
3	2.674	j: multiplicity
4	3.672	λ: eigenvalue
5	4.670	τ_j: delay time
6	5.670	$\bar{\alpha} = \lambda\tau_j$
7	6.669	
8	7.669	
9	8.668	

where $\chi_0 = 1$ and $\chi_p = 0$. The recursive relationship of Eq. 36 can be re-expressed as

$$\chi_k = q_{p-k}\chi_{k-1} \tag{37}$$

where

$$q_1 = \frac{1}{3} \text{ and } q_{p-k} = \frac{1 + q_{p-k-1}}{3 - q_{p-k-1}}, \quad k = 1 \ldots p-1$$

It then follows from Eq. 37 that

$$\chi_k = \prod_{j=p-k}^{p-1} q_j, \quad k = 1 \ldots p-1 \tag{38}$$

Hence,

$$t_k = \frac{-1}{\lambda} \ln \chi_k$$

$$\tau_k = \frac{-1}{\lambda} \ln \left\{ \frac{1}{2} (\chi_{k+1} + \chi_k) \right\} \tag{39}$$

and the weighting constants are defined as

$$a_k = \chi_{k-1} - \chi_k \tag{40}$$

Equations 39 and 40 provide values of τ_k and a_k for the jth element as defined by Eq. 18.

Case II (z_2 defined by Eq. 23)

The approximation in Eq. 21 can be illustrated by Figure 1 in an analogous manner to that discussed in Case Ia. If the IAE criterion is minimized then the values of the time delays are those given in Table 1. If the multiplicity of the eigenvalues is ≥ 2, then

$$\bar{\alpha} = \lambda\tau_j \approx (j-1) + 0.67 \tag{41}$$

If the integral-error (IE) criterion is used (area $S_1 = S_2$ in Figure 1) then

$$\bar{\alpha} = \lambda\tau_j = j, \quad j = 1,2 \ldots n_2 \tag{42}$$

Case III (z_2 defined by Eq. 26)

Because of the complex conjugate eigenvalues, the responses corresponding to $g(s)$ in Eq. 24 are oscillatory rather than overdamped as shown in Figure 1. However, the same general approach can be used to determine suitable values for the time delay elements. Value of τ_α and τ_β in Eq. 25 can be determined using Figure 3, which was calculated by minimizing the integral of the square of the error (ISE).

APPLICATIONS OF REDUCED ORDER MODELS

Open Loop Response of a Pilot Plant Evaporator

The pilot plant evaporator at the University of Alberta has been the subject of previous applications of reduced order models and reduced order control laws (cf. Wilson, 1974, and Wilson et al., 1972, 1973, 1974). Numerous models of the evaporator have been developed (e.g. Fisher and Seborg, 1976), but the most widely used is the fifth order state space model:

$$\dot{x} = Ax + Bu \tag{43}$$

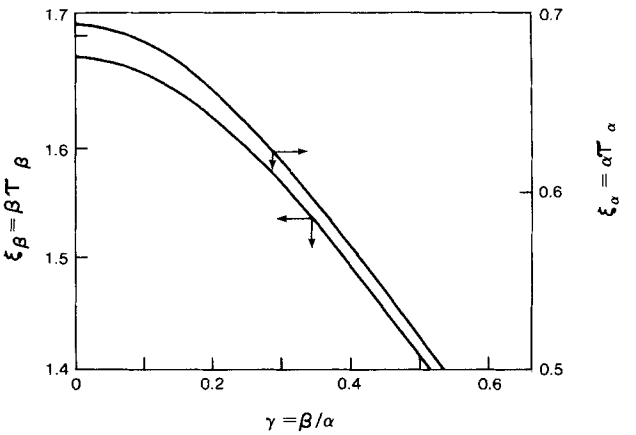

Figure 3. Time delay values that minimize the integral of the square of the error (ISE) for Case III (complex roots $-\alpha \pm \beta i$).

The coefficient matrices for Eq. 43 are given in Appendix 1, and the eigenvalues are 0, 0, −3.796E-2, −7.658E-2 and −7.729E-1. A third order model for the evaporator is convenient because then there are three *output* variables (equal to the three states) and three input variables. Application of the model reduction techniques discussed in this paper led to the following third order models:

a) Marshall's method (MA)

$$\dot{x}_1(t) = A_R x_1(t) + B_R u(t) \tag{44}$$

b) Exact time delay model (ETDM)

$$\dot{x}_1(t) = A_R x_1(t) + B_{R0} u(t) + B_{R1} u(t - \tau_1) + B_{R2} u(t - \tau_2) \tag{45}$$

c) Single (approximate) time delay model (ATDM)

$$\dot{x}_1(t) = A_R x_1(t) + B_{R0} u(t) + B_{RA} u(t - \tau_A) \tag{46}$$

d) Multiple (two-stage) time delay model (MTDM)

$$\dot{x}(t) = A_R x_1(t) + B_{R0} u(t) + B_{R1} \sum_{k=1}^{2} a_{1k} u(t - \tau_{1k}) + B_{R2} \sum_{k=1}^{2} a_{2k} u(t - \tau_{2k}) \tag{47}$$

The coefficient matrices are given in Appendix 1. The state vector in all of the third order models consists of liquid level #1, liquid level #2, and the product concentration (first effect enthalpy and concentration were eliminated). Note also that all of these models belong to Case I (distinct eigenvalues) and

a) Matrix A_R is the same in all models
b) $B_{R0} + B_{R1} + B_{R2} = B_R$
c) $B_{R0} + B_{RA} = B_R$

or, in other words, that all of the models reduce to the one obtained by Marshall's method, when the time delays are set to zero.

The simulated, open-loop responses of the reduced order models to a 10% change in evaporator feed rate, u_1, are shown in Figure 4. The values plotted are the *difference* between the response of the fifth order and the third order model. Thus, the zero-error line can be used as an indicator of the perfect response. The following conclusions follow from Figure 4 and are supported by other unpublished data:

1) Except for the larger overshoot with the single average time delay (ATDM), all of the models with time delays gave a more accurate response than Marshall's approach.

2) Inclusion of two time delays (MTDM) resulted in less error than the models with one time delay calculated using an average, or the maximum input sensitivity approach (ATDM).

3) The use of multiple time delays (MTDM), i.e., two in each control variable, resulted in the best approximation.

All of these conclusions are what one would expect based on an intuitive analysis.

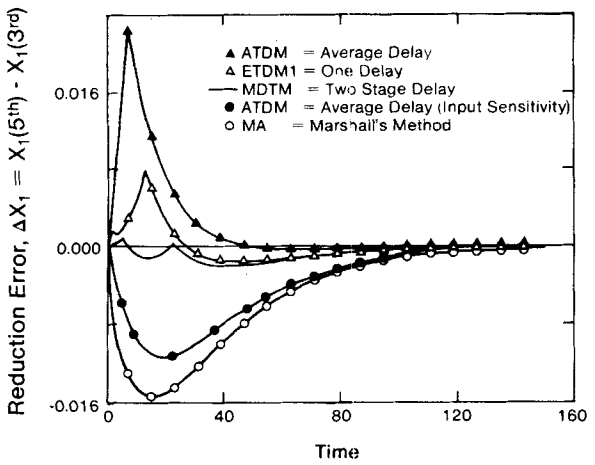

Figure 4. Simulated open-loop step response of different models of the evaporator, showing difference between original 5th order model and reduced 3rd order models.

Closed Loop Response

As shown in the previous section, the inclusion of time delays in the control variables can improve the accuracy of the reduced order models. However, one of the disadvantages is that many of the design techniques for feedback controllers are not applicable to systems with time delays. The purpose of the following example was to show that the model reduction techniques presented in this paper can result in better closed loop responses than Marshall's original method.

The following high order model (HM) was selected so that the calculations would be easy to follow:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-0.2 & 1 & 1\\ 0 & -0.4 & 1\\ 0 & 0 & -0.5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\\ b & b\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{48}$$

The following two second order models were then calculated:

(i) Marshall's model (MA):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-0.2 & 1\\ 0 & -0.4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1+2b & 2b\\ 2b & 1+2b\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{49}$$

(ii) Exact time-delay model (ETDM):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-0.2 & 1\\ 0 & -0.4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + 2b\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}u_1(t-t_D)\\ u_2(t-t_D)\end{bmatrix} \tag{50}$$

where $t_D = 2$.

The following decoupling control laws were then designed using the method of Tsafestas and Paraskevopouros (1973).

$$\text{HM: } u_{HM} = F_{HM}x + G_{HM}v \tag{51}$$

$$\text{MA: } u_{MA} = F_{MA}x_1 + G_{MA}v \tag{52}$$

$$\text{ETDM: } u_{TD} = F_{TD}x_1 + G_{TD}v + K_{TD}u_{TD}(t-\tau) \tag{53}$$

where $v = [v_1, v_2]^T$ is the reference input vector (setpoints). The coefficient matrices for the control laws, Eqs. 51-53 are:

$$F_{HM} = -\begin{bmatrix}-0.2+\lambda_1 & 1 & 1\\ 0 & -0.4+\lambda_2 & 1\end{bmatrix},\ G_{HM} = \begin{bmatrix}f_1 & 0\\ 0 & f_2\end{bmatrix}$$

$$F_{MA} = -\begin{bmatrix}1+2b & 2b\\ 2b & 1+2b\end{bmatrix}^{-1}\begin{bmatrix}-0.2+\lambda_1 & 1\\ 0 & -0.4+\lambda_2\end{bmatrix},\quad b \neq -0.25$$

$$G_{MA} = \begin{bmatrix}1+2b & 2b\\ 2b & 1+2b\end{bmatrix}^{-1}\begin{bmatrix}f_1 & 0\\ 0 & f_2\end{bmatrix}$$

$$F_{TD} = -\begin{bmatrix}-0.2+\lambda_1 & 1\\ 0 & -0.4+\lambda_2\end{bmatrix},$$

$$G_{TD} = G_{HM},\ K_{TD} = -2b\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

The parameters λ_1, λ_2, f_1 and f_2 are selected by the designer and the ideal input/output response of the decoupled system is:

$$\dot{y}_i = -\lambda_i y_i + f_i v_i \qquad i = 1,2 \tag{54}$$

Again for simplicity, it was assumed that $\lambda_1 = \lambda_2 = f_1 = f_2 = \lambda$.

When control laws designed on the basis of reduced order models are applied to the original high order system there is no guarantee of perfect decoupling, or even of stability. Therefore, these performance characteristics must be checked separately.

The stability of the closed loop system composed of the high order model, Eq. 48, and the control law without time delays, Eq. 49, can be determined using Hurwitz stability analysis. The system is stable for $\lambda > 0$ and $b > -0.25$.

The Hurwitz criterion cannot be used to determine the stability of systems with time delays in the control variables. Other stability analysis techniques, such as the characteristic loci method described by MacFarlane (1972) and Lee (1976), were applied to this example but are not described here.

Figure 5 shows the output response of the closed loop systems using the u_{MA} and u_{TD} control laws (Eqs. 52 and 53). The reference input was a step $v = [0,1]^T$ with $\lambda = 0.11$ and $b = 0.2$. Perfect decoupling is not achieved with either of the reduced order controllers. However the inclusion of time delays improves the performance of the reduced order controller considerably.

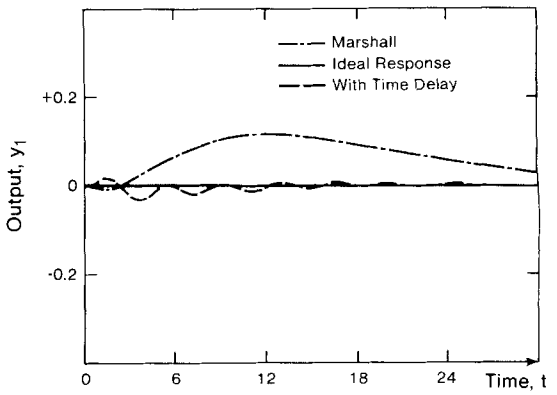

Figure 5a. Simulated closed-loop response of a decoupled system to a unit step change in the setpoint of y_2.

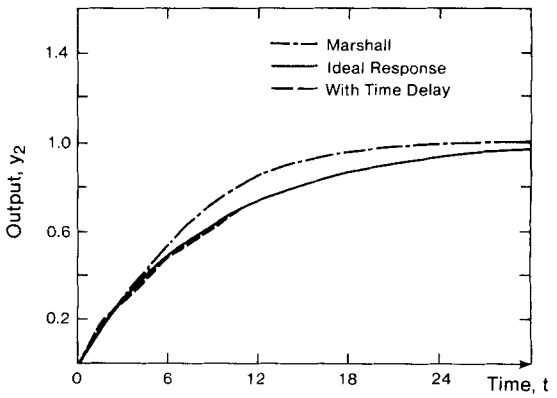

Figure 5b. Simulated closed-loop response of a decoupled system to a unit step change in the setpoint of y_2.

In an actual application it would be possible to "field tune" the values of τ in Eq. 53. Careful consideration should also be given to the relative difficulty of implementing the different controllers on an application. For instance, Eqs. 51 and 53 are probably equally difficult to implement and hence there is little practical advantage for this simple example.

NOTATION

A,B,C = coefficient matrices of state space model
G = transfer function matrix
M = modal matrix of A
V = matrix = $V = M^{-1}$
g = an element of the transfer function matrix G
n = order of high order model, $n = n_1 + n_2$
n_1 = order of low (reduced) order model
n_2 = number of state variables omitted from reduced order model
s = Laplace transform variable
u = vector of r input variables
x = vector of n state variables $x = [x_1,x_2]^T$
z = from similarity transformation $z = M^{-1}x$ (Eq. 2)

Greek Letters

α,β = define complex number $\alpha + \beta i$
χ = $e^{-\lambda t}$ (Eq. 36)
Γ = coefficient matrix of inputs, u (Eq. 5)
λ = eigenvalue of A
τ = time delay, $e^{-\tau s}$, or time constant
Λ = matrix of eigenvalues of A (Eqs. 10–12)

APPENDIX 1

$$A = \begin{bmatrix} 0 & 0 & 0 & -1.085\text{E-}3 & -1.254\text{E-}1 \\ 0 & 0 & 1.258\text{E-}4 & -1.219\text{E-}3 & -1.447\text{E-}1 \\ 0 & 0 & -3.796\text{E-}2 & 3.930\text{E-}2 & 1.447\text{E-}1 \\ 0 & 0 & 0 & -7.550\text{E-}2 & 1.254\text{E-}1 \\ 0 & 0 & 0 & -6.036\text{E-}3 & -7.740\text{E-}1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -7.658\text{E-}2 & 0 \\ 0 & 7.945\text{E-}2 & -3.808\text{E-}2 \\ 0 & -4.135\text{E-}2 & 0 \\ 0 & 0 & 0 \\ 2.159\text{E-}1 & 0 & 0 \end{bmatrix}$$

Third-Order Reduced Model

$$A_R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1.258\text{E-}4 \\ 0 & 0 & -3.796\text{E-}2 \end{bmatrix}$$

$$B_R = \begin{bmatrix} -3.503\text{E-}2 & -7.658\text{E-}2 & 0 \\ -4.040\text{E-}2 & 7.945\text{E-}2 & -3.808\text{E-}2 \\ 5.786\text{E-}2 & -4.135\text{E-}2 & 0 \end{bmatrix}$$

$$B_{R0} = \begin{bmatrix} 0 & -7.658\text{E-}2 & 0 \\ 0 & 7.945\text{E-}2 & -3.808\text{E-}2 \\ 0 & -4.135\text{E-}2 & 0 \end{bmatrix}$$

$$B_{R1} = \begin{bmatrix} 1.600\text{E-}7 & 0 & 0 \\ 1.700\text{E-}5 & 0 & 0 \\ 1.936\text{E-}2 & 0 & 0 \end{bmatrix}$$

$$B_{R2} = \begin{bmatrix} -3.503\text{E-}2 & 0 & 0 \\ -4.040\text{E-}2 & 0 & 0 \\ 3.850\text{E-}2 & 0 & 0 \end{bmatrix}$$

$$B_{RA} = \begin{bmatrix} -3.503\text{E-}2 & 0 & 0 \\ -4.040\text{E-}2 & 0 & 0 \\ 5.786\text{E-}2 & 0 & 0 \end{bmatrix}$$

- EDTM

$$\tau_1 = (7.658\text{E-}2)^{-1} \approx 13 \text{ (min) (IE)}$$

$$\tau_2 = (7.729\text{E-}1)^{-1} \approx 1.3 \text{ (min) (IE)}$$

- MDTM

$$\tau_{11} = 5.294 \text{ (min)}, \tau_{12} = 2.339\text{E1 (min) (IE)}$$

$$\tau_{21} = 5.246\text{E-}1 \text{ (min)}, \tau_{22} = 2.318 \text{ (min) (IE)}$$

$$a_{11} = a_{21} = \tfrac{2}{3}, a_{12} = a_{22} = \tfrac{1}{3}$$

- ATDM

$$\tau_A = \frac{1}{2}(T_{D1} + T_{D2}) \approx 7.2 \text{ (min) (Averaging Time-Delay)}$$

$$\tau_A = \tau_2 \approx 1.3 \text{ (min) (Maximum Input Sensitivity)}$$

$$\left(\frac{\Delta z(\infty)}{\delta} \approx 0.032, \frac{\Delta z_2(\infty)}{\delta} \approx 0.276\right)$$

LITERATURE CITED

Bosley, M. J., and F. P. Lees, "A Survey of Simple Transfer-Function Derivations from High-Order State-Variable Models," *Automatica*, 8(6), 765 (Nov., 1972).

Chen, C. F., "Model Reduction of Multivariable Control System by Means of Matrix Continued Fraction," *Int. J. Control*, **20**, 225 (1974).

Chen, C. F., and L. S. Shieh, "A Novel Approach to Linear Model Simplification," *Proc. JACC*, 454 (1968).

Chidambara, M. R., and E. J. Davison, "A Method for Simplifying Linear Dynamic Systems," *IEEE Trans. on Auto. Control*, **AC-12**, 119, 213, 799 (1967).

Davison, E. J., "A Method for Simplifying Linear Dynamic Systems," *IEEE Trans. on Auto. Control*, *AC-11*, 93 (1966).

Decoster, M., and A. R. Van Cauwenberghe, "A Comparative Study of Different Reduction Methods:" Part 1, *Journal A*, **17**(2), 68 (Apr., 1976); Part 2, *Journal A*, **17**(3), (July, 1976).

Edgar, T. F., "Least Squares Model Reduction Using Step Response," *Int. J. Control*, **22**(2), 261 (1975).

Fisher, D. G., and D. E. Seborg, Editors, *Multivariable Computer Control—A Case Study*, North-Holland/American Elsevier (1976).

Graham, E. U., "Simplification of Dynamic Models." Ph.D. Thesis, Carnegie Inst. Tech. (1968).

Hsia, T. C., "On the Simplification of Linear Systems," *IEEE Trans. on Auto. Control*, **AC-17**, 372 (1972).

Inoue, A., and Z. Iwai, "Design and Application of Reduced-Order Adaptive Observers with Exponential Rate of Convergence for Single-Input, Single-Output Systems," *Int. J. Control*, **39**(2), 375 (1984).

Iwai, Z., and Y. Kubo, "Determination of Eigenvalues in Marshall's Model Reduction," *Int. J. Control*, **30**(5), 823 (1979).

Jonckheere, E. A., and L. M. Silverman, "A New Set of Invariants for Linear Systems—Application to Reduced Order Compensator Design," *IEEE Trans. on Auto. Control*, **AC-28**(10), 953 (1983).

Kokotovic, P. V., and P. Sannuti, "Singular Perturbation Method for Reducing the Model Order in Optimal Control Design," *IEEE Trans. on Auto. Control*, **AC-13**, 377 (1968).

Kung, S. Y., "A New Identification and Model Reduction Algorithm via Singular Value Decompositions," *Proc. 12th Asilomar Conference Circuits, Syst. Comput.*, 705 (1978).

Lee, E. B., "Stability of Linear Multivariable Feedback Systems with Time-Lags," *Control Theory and Topics in Functional Analysis*, 3, Int. Atomic Energy Agency, Vienna (1976).

MacFarlane, A. G., "A Survey of Some Recent Results in Linear Multivariable Feedback Theory," *Automatica*, **8**, 445 (1972).

Marshall, S. A., "An Approximate Method for Reducing the Order of Linear Systems," *Control*, **10**, 642 (1966).
Meier, L., and D. Luenberger, "Approximation of Linear Constant Systems," *IEEE Trans. on Auto. Control*, **AC-12**, 585 (1967).
Moore, B. C., "Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction," *IEEE Trans. on Auto. Control*, **AC-26**, 17 (1981). (see also earlier *System Control Reports 7,801, 7,802*, Dept. of EE, U. of Toronto, Canada.)
Nicholson, H., "Dynamic Optimization of a Boiler," *Proc. IEE*, **111**, 1,479 (1964).
Parthasarthy, R., K. N. Jayasimha, and S. John, "On Model Reduction by Modified Cauer Form," *IEEE Trnas. on Auto. Control*, **AC-28**(4), 523 (1983).
Riggs, J. B., and T. F. Edgar, "Least Squares Reduction of Linear Systems Using Impulse Response," *Int. J. Control*, **20**(2), 213 (1974).
Shamash, Y., "Model Reduction Using the Routh Stability Criterion and the Pade Approximation Technique," *Int. J. Control*, **21**, 475 (1975).
Sinha, N. K., and G. T. Bereznai, "Optimal Approximation of High Order System by Low Order Model," *Int. J. Control*, **14**, 951 (1971).
Shokoohi, S., L. M. Silverman, and P. M. Van Dooren, "Linear Time-Variable Systems: Balancing and Model Reduction," *IEEE Trans. on Auto. Control*, **AC-28**(8), 810 (1983).
Tsafestas, S. G., and P. N. Paraskevopouros, "On the Decoupling of Multivariable Control Systems with Time Delays," *Int. J. Control*, **17**, 405 (1973).
Tung, L. S., and T. F. Edgar, "Least Squares Model Reduction of Multiple Input-Multiple Output Linear Systems," *Proc. Joint Auto. Control Conf.*, Denver (1979).
Wilson, R. G., D. G. Fisher, and D. E. Seborg, "Model Reduction for Discrete Time Dynamic Systems," *Int. J. Control*, **16**, 549 (1972).
Wilson, R. G., D. E. Seborg, and D. G. Fisher, "Model Approach to Control Law Reduction," *Proc. Joint Auto. Control Conf.*, Columbus, OH, 554 (1973).
Wilson, R. G., "Model Reduction and the Design of Reduced Order Control Laws," Ph.D. Thesis, Univ. of Alberta (1974).
Wilson, R. G., D. G. Fisher, and D. E. Seborg, "Model Reduction and the Design of Reduced Order Control Laws," *AIChE J.*, **20**(6), 1,131 (1974).
Wilson, R. G., D. E. Seborg, and D. G. Fisher, "A Least Squares Approach to Model Reduction and the Design of Low-Order Controllers," *Can. J. Chem. Eng.*, **54**, 220 (June, 1976).

Manuscript received April 10, 1979; revision received April 17, 1984, and accepted April 30.

Taylor-Aris Dispersion Arising from Flow in a Sinusoidal Tube

An analysis of dispersion arising from creeping flow through a periodically constricted tube is developed with the goal of evaluating the effect of converging and diverging pore structure on dispersion in porous media. The theory is based on rigorous long-time solutions of the convective-diffusion equation. The calculations are done numerically over a range of Peclet numbers and tube geometries; for the geometrical parameters which correlate permeability data in packed beds, the results of the present work agree well with liteature data for dispersion in such beds.

D. A. HOAGLAND and R. K. PRUD'HOMME

Department of Chemical Engineering
Princeton University
Princeton, NJ 08544

SCOPE

Fluid motion in periodically constricted tubes has been studied by several authors interested in modeling flow through porous media. Using a model which replaces a porous medium with an equivalent array of periodically constricted tubes, these authors have been able to use flow-rate/pressure-drop relationships in the simpler geometry to successfully predict permeabilities of complex porous media. The dispersion of solute during flow through porous materials can be studied using the same geometrical models. The motivation for such modeling of dispersion lies in the crucial role of this phenomenon in oil recovery problems, reactor design, and chromatography. The present work will examine dispersion during creeping flow in a periodically constricted tube; it is proposed that the solution to this problem offers considerable insight into the dispersion processes observed during flow in more complex porous media.

The present work employs the concept of local and global coordinates which was developed by Brenner (1980) to describe transport in periodic structures and is an extension of earlier work of Taylor (1953) and Aris (1956). Using Brenner's analysis, this paper will describe the calculation of dispersion coefficients in sinusoidal tubes for wide ranges of Peclet number and tube geometry. Additionally, analytical solutions have been found to the limiting case of very long wavelength. The numerically determined dispersion coefficients can be compared to literature data for dispersion in packed beds if suitable geometrical parameters are selected. This comparison will test the hypothesis that dispersion in packed beds can be predicted with the same pore structure models used for correlating permeability data. It should be noted that transfer of solute between different flow channels has not been incorporated into the present model.

CONCLUSIONS AND SIGNIFICANCE

Dispersion coefficients for creeping flow in sinusoidal tubes have been calculated both analytically and numerically over wide ranges of tube geometry and Peclet number. The results have been obtained rigorously from long-time solutions of the